CHAPTER

2 APPLICATIONS OF DIFFERENTIATION

Syllabus coverage

TOPIC 3.1: FURTHER DIFFERENTIATION AND APPLICATIONS

The second derivative and applications of differentiation

3.1.10 use the increments formula: $\delta y \approx \frac{dy}{dx} \times \delta x$ to estimate the change in the dependent variable y resulting from changes in the independent variable x

3.1.11 apply the concept of the second derivative as the rate of change of the first derivative function

3.1.12 identify acceleration as the second derivative of position with respect to time

3.1.13 examine the concepts of concavity and points of inflection and their relationship with the second derivative

3.1.14 apply the second derivative test for determining local maxima and minima

3.1.15 sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection

3.1.16 solve optimisation problems from a wide variety of fields using first and second derivatives

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2.1 The increments formula

Video playlist
The increments formula

In Year 11 Methods we learnt that the derivative $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$, where δy is a small change (small increment) in y and δx is a small change (small increment) in x.

The increments formula

If δx is small, we can say $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$.

Rearranging gives $\delta y \approx \frac{dy}{dx} \times \delta x$, which is called the **increments formula**.

$\delta y \approx \frac{dy}{dx} \times \delta x$ for small values of δx.

WORKED EXAMPLE 1 Using the increments formula

Consider the function $y = 3x^3 - x + 1$. Using the increments formula, determine the approximate change in y when x changes from 2 to 2.02.

Steps	Working
1 Determine $\frac{dy}{dx}$, x and δx.	$\frac{dy}{dx} = 9x^2 - 1$, $x = 2$ and $\delta x = 0.02$
2 Use the increments formula to determine the approximate change in y.	$\delta y \approx \frac{dy}{dx} \times \delta x$ $= (9x^2 - 1) \times 0.02$ $= (9 \times 2^2 - 1) \times 0.02$ $= 0.7$

WORKED EXAMPLE 2 Working with the increments formula

A spherical balloon has a radius of 10 cm. Using the increments formula, determine the change in the volume of the balloon if the radius changes to 9.97 cm.

Steps	Working
1 Write down a formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$
2 Determine $\frac{dV}{dr}$, r and δr.	$\frac{dV}{dr} = 4\pi r^2$, $r = 10$ and $\delta r = -0.03$
3 Use the increments formula to determine the approximate change in volume.	$\delta V \approx \frac{dV}{dr} \times \delta r$ $= (4\pi r^2) \times -0.03$ $= (4\pi \times 10^2) \times -0.03$ $= -37.7\,\text{cm}^3$
4 Comment on change in volume.	The volume of the balloon has decreased by $37.7\,\text{cm}^3$.

EXERCISE 2.1 The increments formula

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1 WORKED EXAMPLE 1 Consider the function $y = \frac{3x-1}{\sqrt{x}+1}$. Using the increments formula, determine the approximate change in y when x changes from 5 to 5.01.

2 WORKED EXAMPLE 2 A spherical ball has a radius of 4 cm. Using the increments formula, determine the change in the volume of the ball if the radius changes to 3.95 cm.

3 Given the function $f(x) = \sqrt{x+1} - 4x^2$, use the increments formula to determine the change in y if x changes from 3 to 3.02.

Calculator-free

4 (3 marks) Given that $\sqrt{9} = 3$, use the increments formula to determine an approximation for $\sqrt{9.01}$.

5 (4 marks) Use the increments formula to determine the change in value of the function $y = \frac{4}{x^2} - \sqrt{x}$ if the x value changes from 1 to 1.01.

Calculator-assumed

6 (3 marks) The side of a square is 8 cm. How much will the area of the square increase if the length of the side increases by 1 mm?

7 (4 marks) A spherical ball is slowly deflating. Use the increments formula to determine the approximate change in the radius of the ball if its surface area changes from 12.20 cm^2 to 12.15 cm^2.

8 (4 marks) The height of a cylinder is 10 cm and the radius is 5 mm. Find the approximate change in volume of the cylinder if the radius changes to 5.03 mm and the height remains the same.

2.2 Second derivative and points of inflection

Video playlist Second derivative and points of inflection

The **second derivative** is the rate of change of the first derivative function. In other words, the derivative of the derivative is called the second derivative.

The second derivative

For a function $y = f(x)$, the first derivative is written as y' or $f'(x)$ or $\frac{dy}{dx}$ and the second derivative as y'' or $f''(x)$ or $\frac{d^2y}{dx^2}$.

WORKED EXAMPLE 3 Finding the second derivative

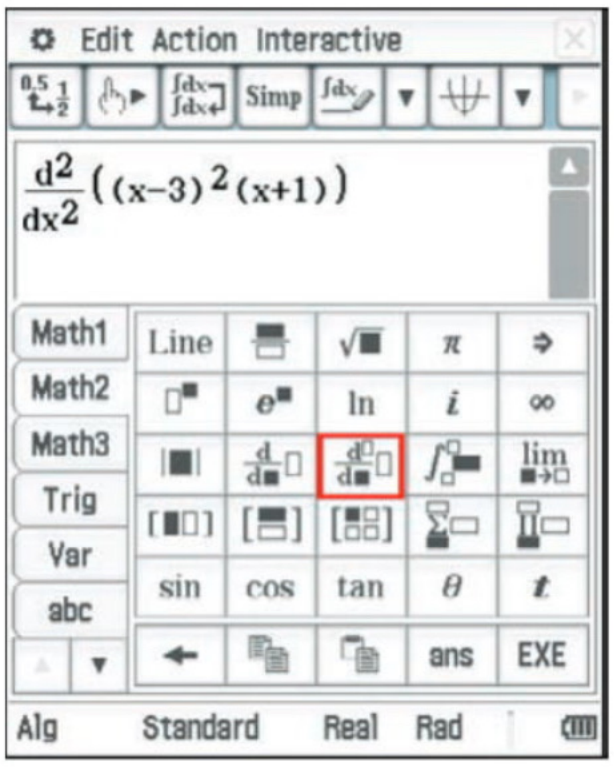
Determine the second derivative of the function $y = (2x^2 - 3)^2$.

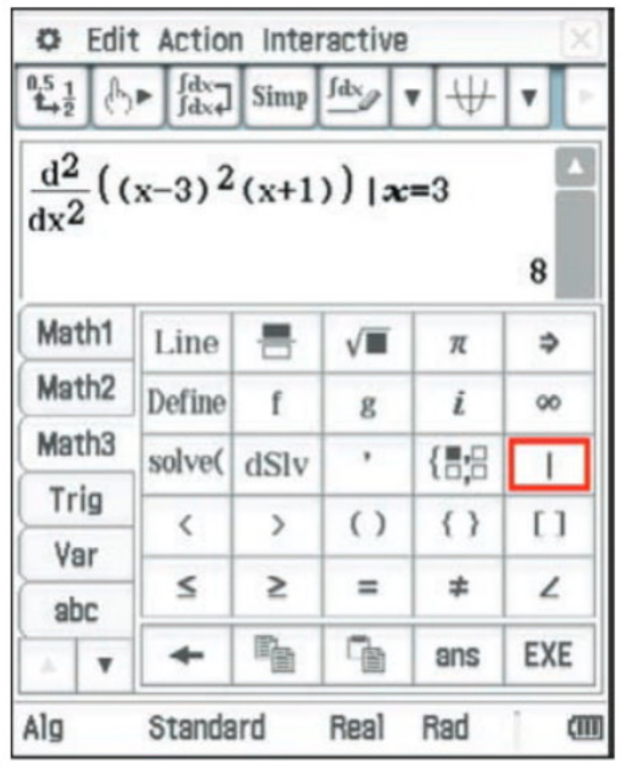
Steps	Working
1 Differentiate once for first derivative.	$y' = 2(2x^2 - 3)4x$ $= 16x^3 - 24x$
2 Differentiate again for second derivative.	$y'' = 48x^2 - 24$

USING CAS 1 Finding the second derivative

Determine the value of the second derivative of the function $f(x) = (x - 3)^2(x + 1)$ at the point where $x = 3$.

ClassPad





1 Enter the expression $(\mathbf{x - 3})^2(\mathbf{x + 1})$.
2 Open the **Keyboard** and tap **Math2**.
3 Tap on the **nth derivative** template.
4 Enter **x** and **2** into the template.
5 Move the cursor to the end of the expression.
6 Tap **Math3**.
7 Tap the | symbol.
8 Enter **x = 3** and press **ENTER**.

Alternatively, tap **Interactive > Calculation diff > Derivative at value**. Complete the fields in the dialogue box and change the **Order** field to **2**.

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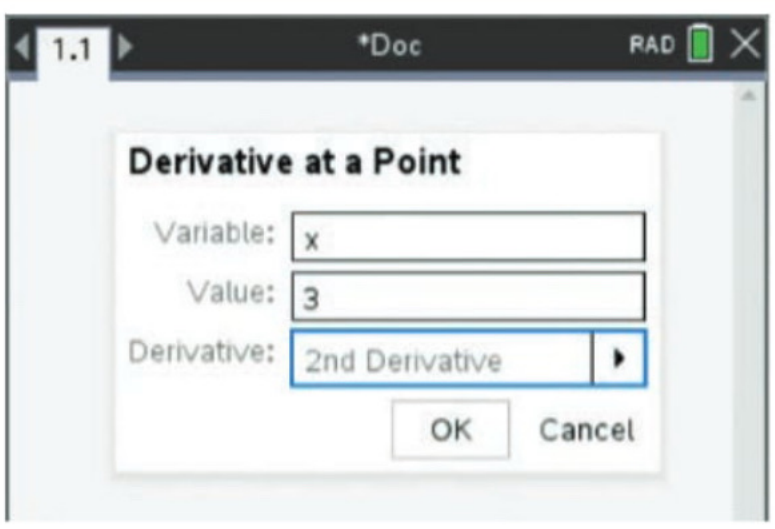
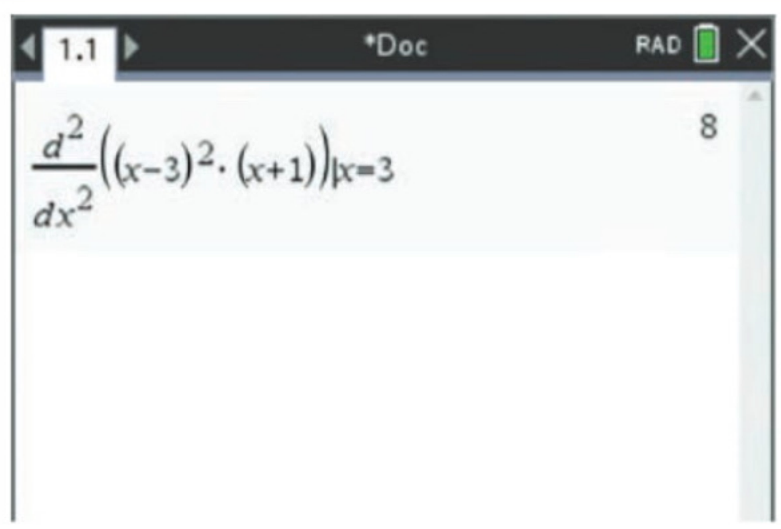
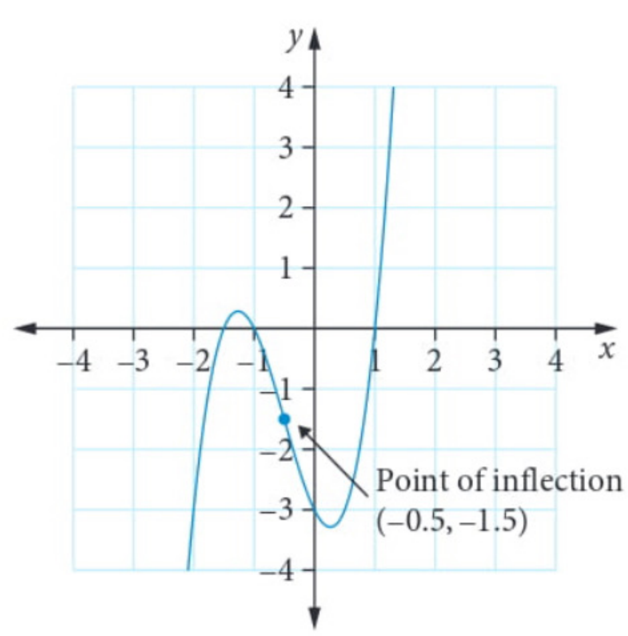




1 Press **menu > Calculus > Derivative at a Point** to open the dialogue box.
2 For the **Value:** field, enter **3**.
3 Change the **Derivative:** field to **2nd Derivative**.
4 Press **OK**.
5 In the derivative template, enter the expression $(\mathbf{x - 3})^2(\mathbf{x + 1})$.
6 Press **enter**.

The value of the second derivative where $x = 3$ is 8.

As with stationary points, a non-stationary **point of inflection** is where a function changes **concavity**. (Stationary points of inflection will be discussed in Section 2.3). For example, for the function $y = (x - 1)(x + 1)(2x + 3)$, as shown on the graph, there is a point of inflection at $(-0.5, -1.5)$. To the left of this point, the graph is concave down and to the right of this point, the graph is concave up.



USING CAS 2 Finding points of inflection using graphs

For the function $y = (x - 3)^2(x + 1)$, describe where the function is concave up and concave down and, hence, state the coordinates of any points of inflection (to two decimal places).

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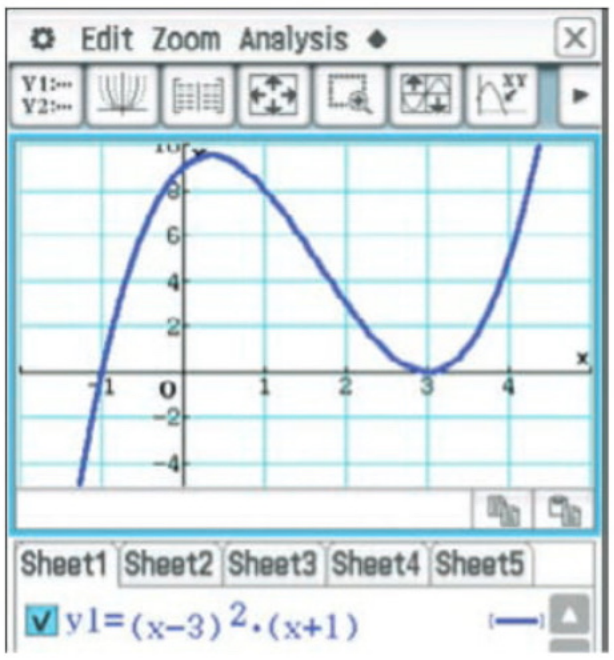

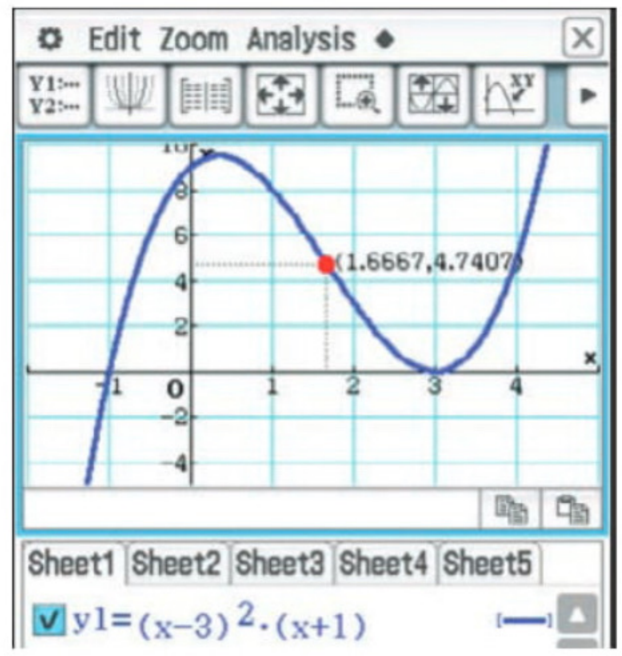

1 Graph the equation $y = (x - 3)^2(x + 1)$.
2 Adjust the window settings to suit.
3 The graph appears to be concave down when x is less than approximately 1.5 and concave up when x is greater than approximately 1.5.
4 Tap **Analysis > G-Solve > Inflection**.
5 The cursor will jump to the point of inflection.
6 Press **EXE** to paste the coordinates of the point of inflection on the graph.

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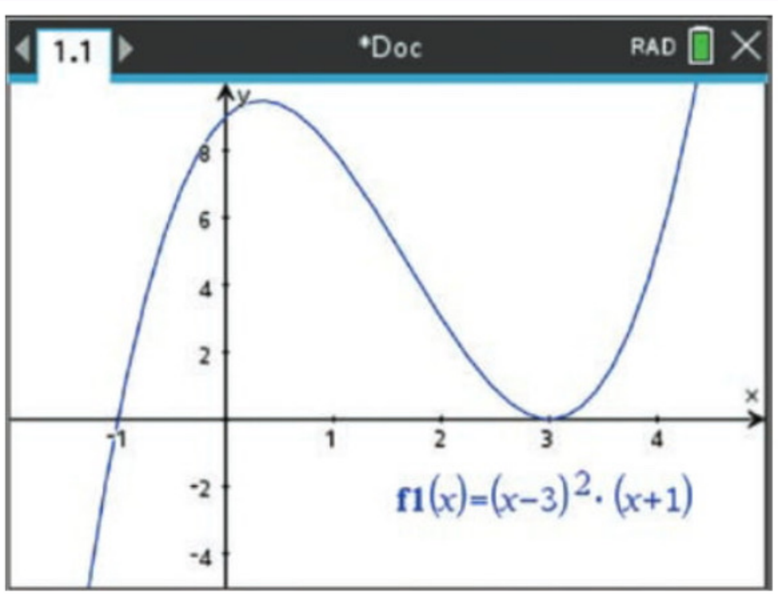

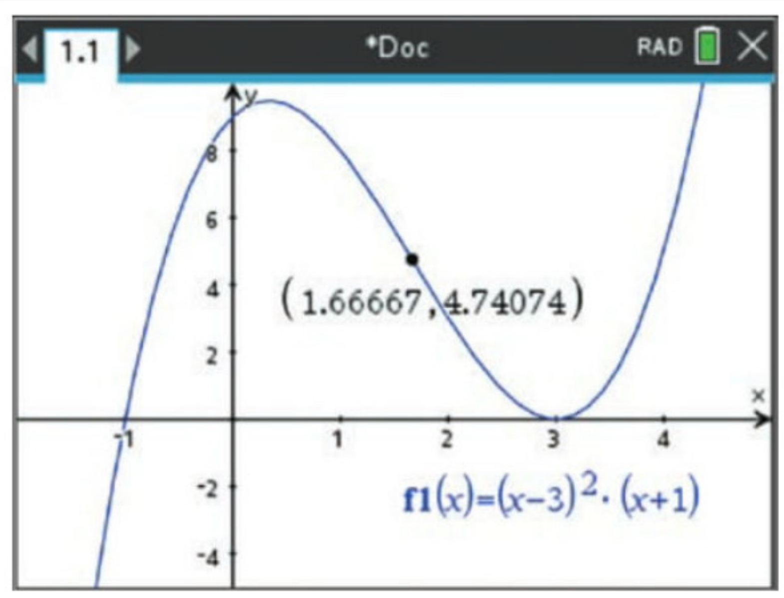

1 Graph the equation $y = (x - 3)^2(x + 1)$.
2 Adjust the window settings to suit.
3 The graph appears to be concave down when x is less than approximately 1.5 and concave up when x is greater than approximately 1.5.
4 Press **menu > Analyze Graph > Inflection**.
5 When prompted for the **lower bound?**, move the cursor to the concave down section and press **enter**.
6 When prompted for the **upper bound?**, move the cursor to the concave up section and press **enter**.
7 The coordinates of the point of inflection will appear on the screen.

Rounding to two decimal places, the graph is concave down for $x < 1.67$ and concave up for $x > 1.67$.
The point of inflection is $(1.67, 4.74)$.

EXERCISE 2.2 Second derivative and points of inflection

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Recap

1 A rectangle is such that the length is three times the width. Determine the change in the perimeter if the width changes from 2 cm to 2.01 cm.

2 If the diameter of a sphere decreases from 25 cm to 24.9 cm, what is the change in the surface area?

Mastery

3 WORKED EXAMPLE 3 Determine the second derivative of each of the functions below.

a $y = (x-3)^{\frac{1}{2}}$ **b** $f(x) = \dfrac{x^2-1}{2x+10}$ **c** $y = 3x - 2$

d $f(x) = \dfrac{3}{2}x^{\frac{3}{2}}$ **e** $y = (2x+1)(x^2+3)^3$ **f** $y = 3$

4 Using CAS 1 Determine the value of the second derivative of the function $f(x) = \dfrac{1}{2}(x^2 - 3x + 2)^2$ at the point $x = -2$.

5 Using CAS 2 For the function $f(x) = x(3x-1)^2$, describe where the function is concave up and concave down and, hence, state the coordinates of any points of inflection. Give your answers to two decimal places.

Calculator-free

6 (2 marks) Give an example of a function where $f'(x) = f''(x)$.

7 (1 mark) How many points of inflection does the function $y = (x-3)(x+3)(x-2)$ have?

Calculator-assumed

8 (4 marks) Determine the value of a and b (where a and b are greater than 0) if $f(x) = (x+a)^2(2x-b)$, $f'(2) = 24$ and $f''(2) = 26$.

9 (3 marks) Given the function $y = \dfrac{x}{\sqrt{x^2+2}}$, determine value(s) of x for which $\dfrac{dy}{dx} + \dfrac{d^2y}{dx^2} = 0$.

2.3 Stationary points

2.3

Remember that the gradient of the graph of $y = f(x)$ is $f'(x)$ or $\frac{dy}{dx}$.

A positive gradient points up, from left to right. 

A negative gradient points down, from left to right. 

A function is **increasing** when it has a positive gradient $\left(\frac{dy}{dx} > 0\right)$ and its graph is pointing up.

A function is **decreasing** when it has a negative gradient $\left(\frac{dy}{dx} < 0\right)$ and its graph is pointing down.

Video playlist
Stationary points

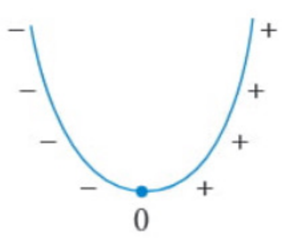
Worksheets
'The sign of the derivative

Stationary points

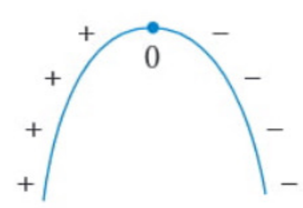
Stationary points

A **stationary point** on the graph of a function is a point where the graph has zero gradient $\left(f'(x) = 0 \text{ or } \frac{dy}{dx} = 0\right)$; that is, the graph is instantaneously flat, neither increasing nor decreasing.

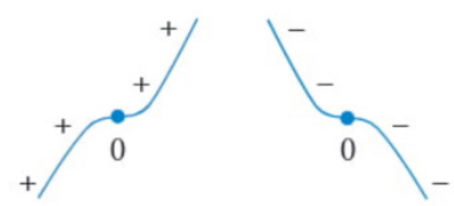
A stationary point is either a **turning point** or a **stationary point of inflection**.



Local minimum turning point



Local maximum turning point



Stationary point of inflection

Turning points

A **turning point** is found where $f'(x) = 0$ and the sign of the gradient changes on either side of the stationary point, from negative to positive (for a minimum point) or from positive to negative (for a maximum point).

Turning points are also called **local maximum** or **local minimum points** because they represent the highest or lowest values of the function in the local vicinity or neighbourhood. They may not actually be the absolute maximum or minimum points for the entire function, which are called the **global maximum** or **global minimum points**.

The second derivative test for finding maximum and minimum points

The second derivative can determine if a turning point is a maximum or minimum.

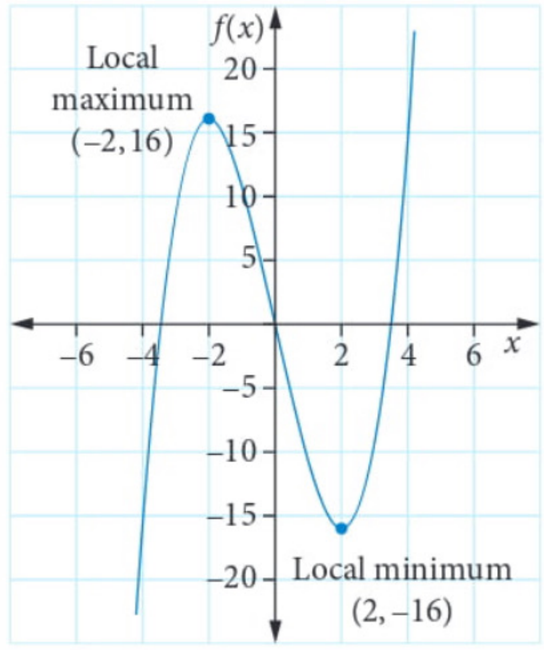
If $f''(x) > 0$, the turning point is a minimum.

If $f''(x) < 0$, the turning point is a maximum.

The function graphed on the right has stationary points at $(-2, 16)$ and $(2, -16)$. At these points, $f'(x) = 0$.

At the point $(-2, 16)$, $f''(x) < 0$, hence it is a maximum point.

At the point $(2, -16)$, $f''(x) > 0$, hence it is a minimum point.



WORKED EXAMPLE 4 Turning points

Find the coordinates and nature of the turning points on the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

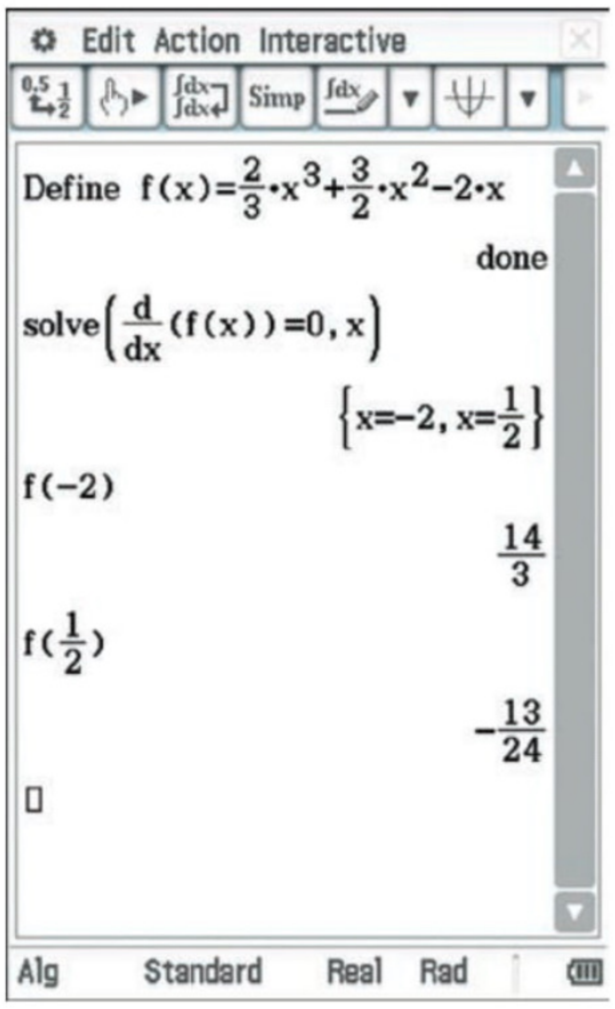
Steps	Working
1 Solve $f'(x) = 0$ for stationary points.	$f'(x) = 2x^2 + 3x - 2 = 0$ $(x + 2)(2x - 1) = 0$ $x = -2$ or $2x = 1$ $x = -2$ or $x = \frac{1}{2}$
2 Find $f''(x)$ and determine if it's positive or negative to identify the nature of the stationary points.	$f''(x) = 4x + 3$ At $x = -2$, $f''(x) < 0$. At $x = \frac{1}{2}$, $f''(x) > 0$.
Alternative method Check whether the sign of the gradient changes on either side to identify the nature of the stationary points. (Also known as the sign test.)	Around $x = -2$, gradient changes from positive to negative. Around $x = \frac{1}{2}$, gradient changes from negative to positive. Both are turning points.
3 Substitute $x = -2$ and $x = -\frac{1}{2}$ into $f(x)$ to determine each y-coordinate.	$f(-2) = \frac{14}{3}$, $f\left(\frac{1}{2}\right) = -\frac{13}{24}$
4 State the coordinates and nature of the turning points.	The coordinates are $\left(-2, \frac{14}{3}\right)$, which is a local maximum, and $\left(\frac{1}{2}, -\frac{13}{24}\right)$, which is a local minimum.

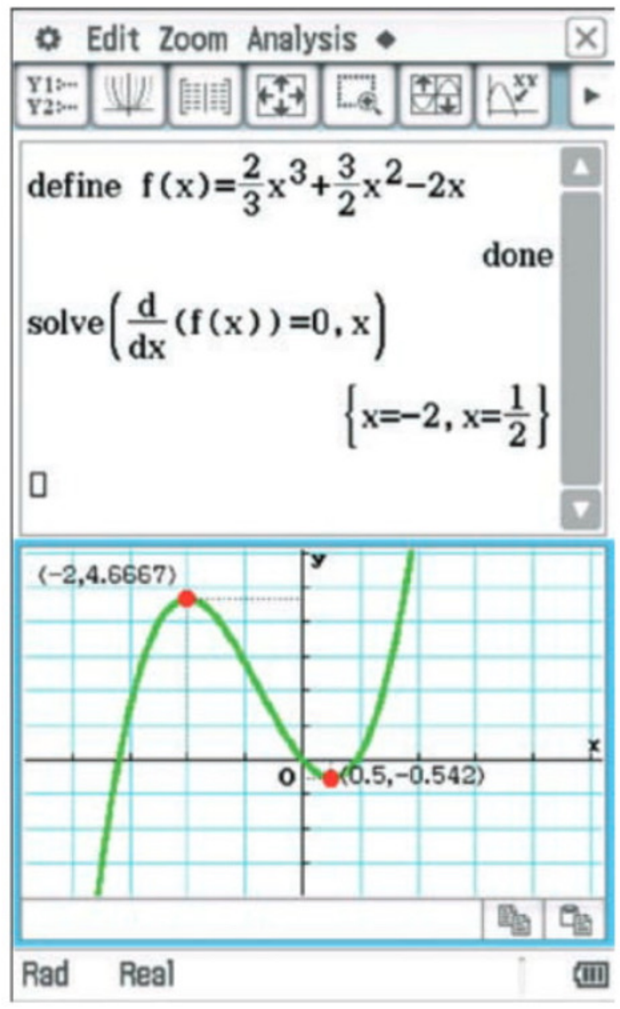
CAS can be used to find the coordinates and nature of stationary points.

USING CAS 3 Finding stationary points

Find the coordinates and nature of the turning points on the graph of $f(x) = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 2x$.

ClassPad





1. Define and highlight **f(x)** as shown above.
2. Derive using **Interactive > Diff**, and equate to zero, then solve using **Equation/Inequality** to solve for x.
3. Substitute the x-coordinates into the defined function to determine the corresponding y-coordinates of the turning points.
4. Graph **f(x)** by dragging into the graph screen.
5. Adjust the window settings to suit.
6. Tap **Analysis > G-Solve > Max** to display the approximate coordinates of the local maximum.
7. Tap **Analysis > G-Solve > Min** to display the approximate coordinates of the local minimum.

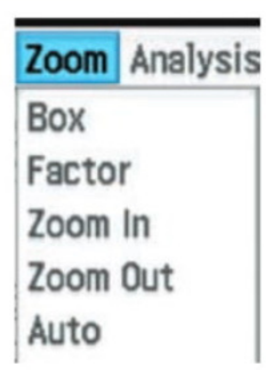
The local maximum point is $\left(-2, \frac{14}{3}\right)$.

The local minimum point is $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

Exam hack

Students using the ClassPad often use Zoom Auto to get a better view of their graph.

The more elegant way to find a view that suits is to use [icon].



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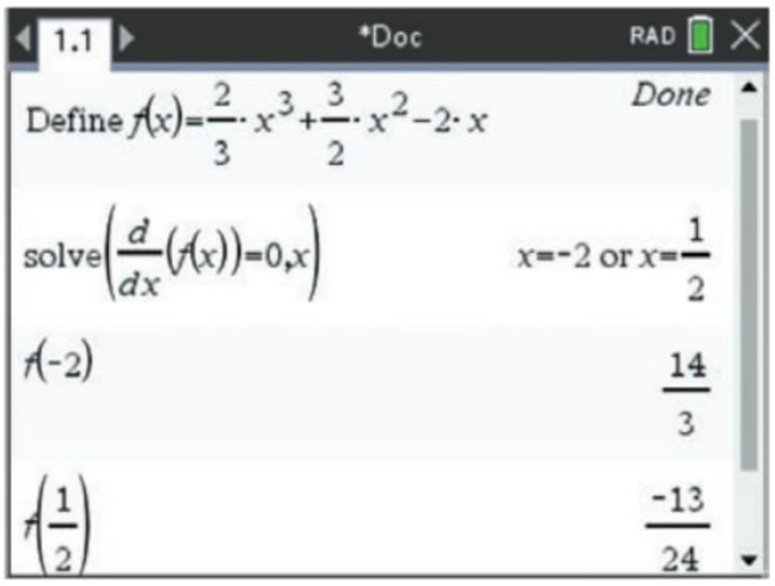
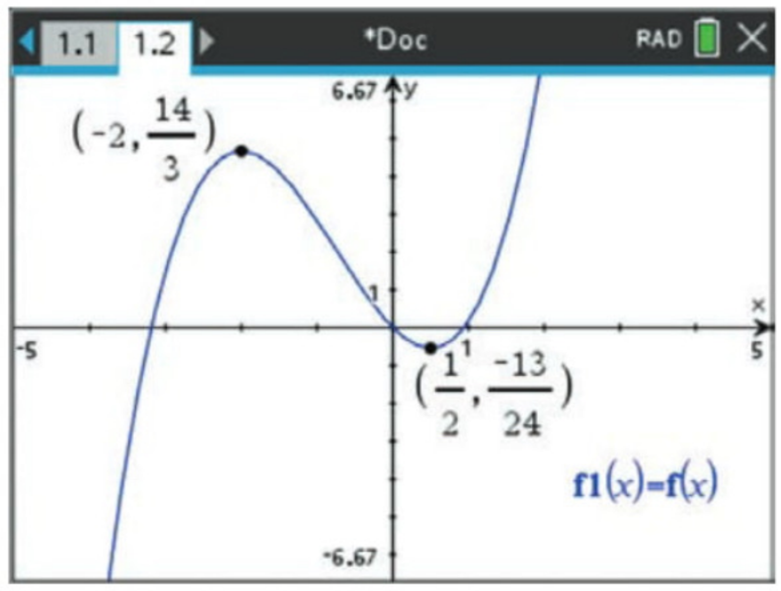




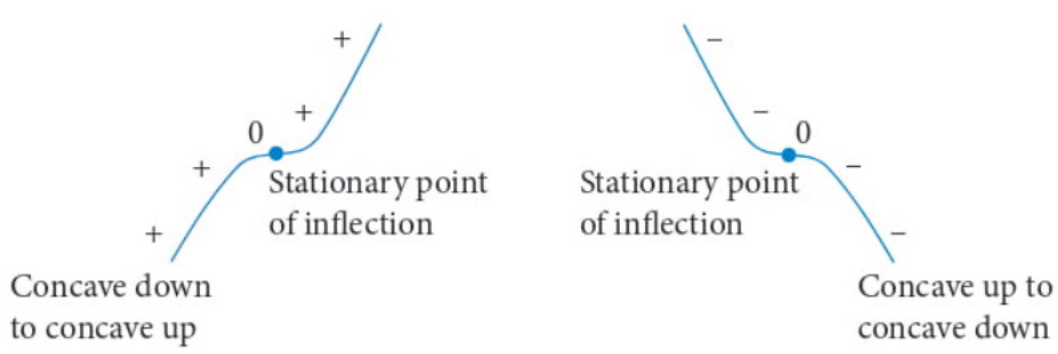
1. Define **f(x)** as shown above.
2. Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.
3. Substitute the x-coordinates into the function to determine the corresponding y-coordinates of the turning points.
4. Graph **f(x)**, which shows the local maximum and minimum.
5. Adjust the window settings to suit.
6. To confirm the turning points, press **menu > Geometry > Points & Lines > Point On**.
7. Click on the graph of the function twice to add two points.
8. Press **esc** to remove the point tool.
9. Click twice on the x-coordinate of the first point and enter **–2**.
10. The exact value of the corresponding y-coordinate will be displayed.
11. Repeat to find the exact coordinates of the second turning point.

The local maximum point is $\left(-2, \frac{14}{3}\right)$.

The local minimum point is $\left(\frac{1}{2}, -\frac{13}{24}\right)$.

Stationary points of inflection

A stationary point of inflection is a flat 'bend' in the graph where $f'(x) = 0$; however, the sign of the gradient stays *the same* on either side of the stationary point.



If $f'(x) = 0$ and $f''(x) = 0$ it cannot be assumed that there is a stationary point of inflection. Further investigation of the gradient on either side of the stationary point is needed.

WORKED EXAMPLE 5 Finding stationary points of inflection

Find the coordinates of any stationary points of inflection on the graph $f(x) = (x+1)^3(x-2)$.

Steps	Working
1 Find $f'(x)$ and solve for zero.	$f'(x) = 4x^3 + 3x^2 - 6x - 5$ When $f'(x) = 0$, then $x = -1$ and $x = \frac{5}{4}$.
2 Find $f''(x)$ and determine the value to identify the nature of the stationary points.	$f''(x) = 12x^2 + 6x - 6$ $f''(-1) = 0$ (possible stationary point of inflection) $f''\left(\frac{5}{4}\right) > 0$ (not a stationary point of inflection)
3 Check to see if the sign of the gradient changes or stays the same on either side to identify the nature of the stationary points.	On either side of $x = -1$, the gradient remains negative. The stationary point of inflection is at $x = -1$.
4 Find the coordinates of the stationary point of inflection.	$f(-1) = 0$ The stationary point of inflection is at $(-1, 0)$.

Stationary points

$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	$f''(x)$ at the stationary point	Type of stationary point
+	0	−	Less than zero	Local maximum point 
−	0	+	Greater than zero	Local minimum point 
+	0	+	0	Stationary point of inflection 
−	0	−	0	Stationary point of inflection 

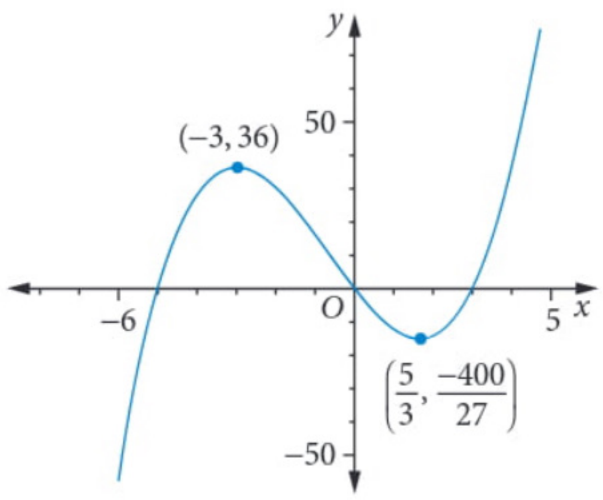
EXERCISE 2.3 Stationary points

ANSWERS p. 389

Recap

1 Let $f(x) = \frac{x^2}{x+1}$. Evaluate $f''(2)$.

2 For the graph shown below, give the approximate coordinates of any inflection points.



Mastery

3 WORKED EXAMPLE 4 Find the coordinates and nature of the turning point on the graph of $f(x) = x^2 - 4x$.

4 WORKED EXAMPLE 5 Find the coordinates of the stationary point of inflection on the graph of $f(x) = 3(x-1)^3(x+2)$.

5 Using CAS 3 State the coordinates and nature of the turning point on the graph with the rule $f(x) = -\frac{1}{2}x^2 - x + 2$.

6 State the local maximum and the stationary point of inflection, respectively, on the graph of $y = -\frac{1}{2}x^4 + \frac{1}{2}x^3 + \frac{3}{2}x^2 - \frac{5}{2}x + 1$.

Calculator-free

7 (4 marks) State any stationary point(s) and their nature for the function $y = \frac{1}{3}x^3 + x^2 - 3x + 1$.

8 (2 marks) What type of stationary point exists at $(1, 1)$ on the graph of the function with rule $y = (x-1)^3 + 1$?

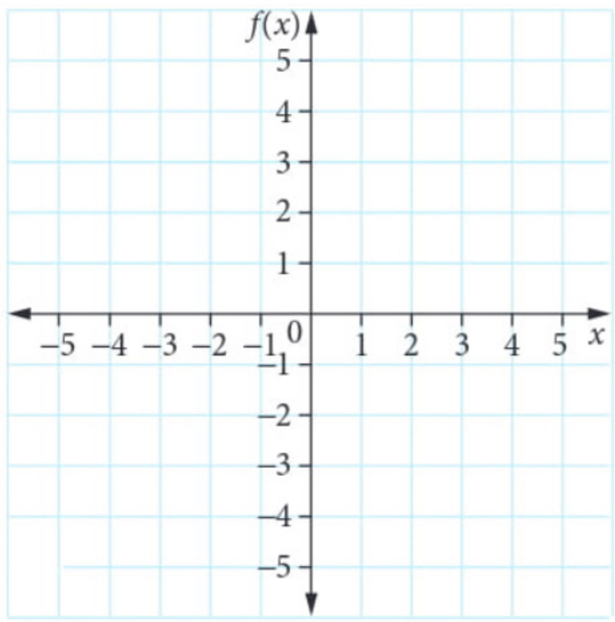
9 (1 mark) Let $f(x)$ be a function such that $f'(3) = 0$ and $f'(x) < 0$ when $x < 3$ and when $x > 3$. Describe what type of stationary point (if any) exists at $x = 3$.

10 (2 marks) A cubic function has the rule $y = f(x)$. The graph of the derivative function $f'(x)$ crosses the x-axis at $(2, 0)$ and $(-3, 0)$. The maximum value of the derivative function is 10. State the value of x for which the graph of $y = f(x)$ has a local maximum.

11 (6 marks) Consider the function $f(x) = 3x^2 - x^3$.

a Find the coordinates and nature of the stationary points of the function. (4 marks)

b Copy the axes below and on it sketch the graph of $f(x)$. (2 marks)



2.3

Exam hack

Sketch graphs with smooth curves and not V-shapes at the turning points.

12 (5 marks) A function is such that

- $f'(x) = 0$ at $x = 0$ and $x = 2$
- $f'(x) < 0$ for $0 < x < 2$ and $x > 2$
- $f''(x) < 0$ at $x = 0$.

State whether each of the statements below is true or false.

a The graph of $f(x)$ has a stationary point of inflection at $x = 0$.

b The graph of $f(x)$ has a local maximum point at $x = 2$.

c The graph of $f(x)$ has a stationary point of inflection at $x = 2$.

d The graph of $f(x)$ has a local maximum point at $x = 0$.

e The graph of $f(x)$ has a local minimum point at $x = 2$.

13 (3 marks) The function $f(x) = x^3 + ax^2 + bx$ has a local minimum at $x = 1$ and a local maximum at $x = -3$. Determine the values of a and b.

Calculator-assumed

14 (6 marks)

a Consider the function $y = 2\sqrt{x} - x^2 + x$. Determine the first derivative. (2 marks)

b Use your result from part **a** to show why there is a stationary point at $x = 1$. (2 marks)

c Find the second derivative and use this to describe the nature of the stationary points at $x = 1$. (2 marks)

Exam hack

When a question asks you to 'show' something, you cannot just write the answer from CAS, you need to show some working.

15 (2 marks) Use calculus to show why the function $f(x) = \sqrt{x} + x^2 + 1$ has no stationary points.

16 (2 marks) Explain why the cubic function $f(x) = ax^3 - bx^2 + cx$, where a, b and c are positive constants, has no stationary points when $c > \dfrac{b^2}{3a}$.

17 (2 marks) Consider the function $f(x) = 4x^3 + 5x - 9$.

a Find $f'(x)$. (1 mark)

b Explain why $f'(x) \geq 5$ for all x. (1 mark)

Exam hack

We need to practise how to answer these one-line 'explain' questions. Referring to the relevant graph is a good strategy. Just saying 'it was translated 5 units up' is not enough detail.

18 (2 marks) The cubic function p is defined by $p(x) = ax^3 + bx^2 + cx + k$, where a, b, c and k are real numbers. If p has m stationary points, what are the possible values of m?

2.4 Curve sketching

Video playlist Curve sketching

Worksheets Curve sketching 2

Further curve sketching

Curve sketching with derivatives

Key features of a graph

When sketching a graph, it is important to identify and show its key features, such as:

- the general shape
- y-intercept
- x-intercept(s)
- stationary points, including their nature.

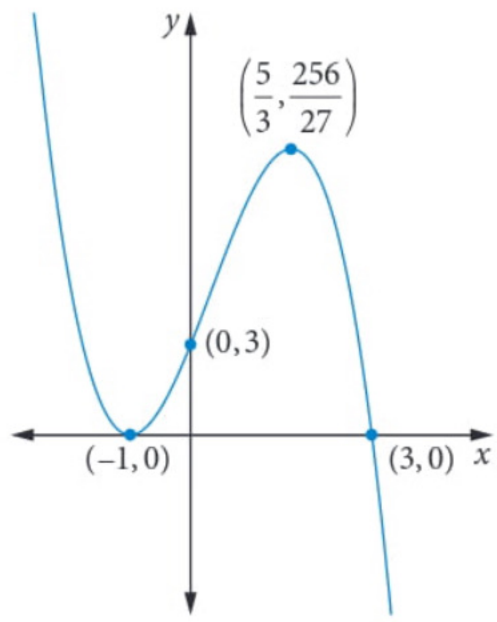
Exam hack

Don't just rely on CAS for graphing. It is important to have a general idea of the shape of the graph and its significant features for calculator-free questions.

WORKED EXAMPLE 6 Sketching a function

Sketch the graph of $f(x) = -(x + 1)^2(x - 3)$, labelling the key features.

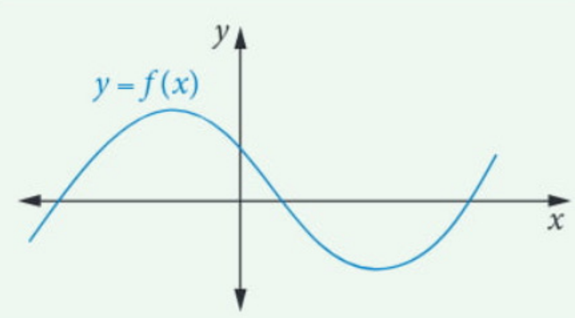
Steps	Working
1 Consider the general shape.	When expanded, the leading coefficient is $-1 < 0$, so the function is an 'upside-down' cubic. 
2 Find the y-intercept.	$f(0) = -(0 + 1)^2(0 - 3) = 3$ y-intercept is 3.
3 Find the x-intercept(s).	$f(x) = -(x + 1)^2(x - 3) = 0$ x-intercepts are -1 and 3.
4 Solve $f'(x) = 0$ for stationary points.	$f'(x) = -(x + 1)^2(1) + (x - 3) \times -2(x + 1)$ $= -(x^2 + 2x + 1) + -2(x^2 + x - 3x - 3)$ $= -x^2 - 2x - 1 - 2x^2 + 4x + 6$ $= -3x^2 + 2x + 5$ Solve $-(3x^2 - 2x - 5) = 0$: $-(3x - 5)(x + 1) = 0$ $x = -1,\ x = \frac{5}{3}$

5 Calculate the y values of the stationary points.	$f(-1) = 0$ and $f\left(\frac{5}{3}\right) = \frac{256}{27}$ Stationary points are $(-1, 0)$ and $\left(\frac{5}{3}, \frac{256}{27}\right)$.
6 Find $f''(x)$ and determine if it's positive or negative to identify the nature of the stationary points.	$f''(x) = -6x + 2$ $f''\left(\frac{5}{3}\right) < 0$, hence a local maximum. $f''(-1) > 0$, hence a local minimum.
7 Sketch the graph.	

The relationship between a function and its derivative can be shown using a graph. We can explore significant points when comparing both the function graph and the derivative graph on the same axes.

WORKED EXAMPLE 7 Graphing derivative functions

The graph of $y = f(x)$ is shown. Sketch the graph of $y = f'(x)$.

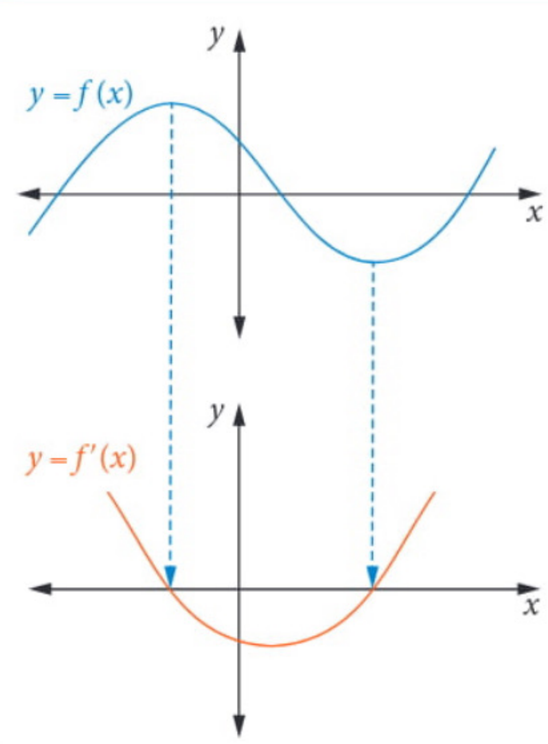


Steps

- The function is increasing at first, so the derivative graph is positive (above the x-axis).
- There is a stationary point just before the y-axis, so the derivative graph is zero (x-intercept).
- Then the function is decreasing, so the derivative graph is negative (below the x-axis).
- At the second stationary point, the derivative graph is zero again.
- Then the function is increasing again, so the derivative graph is positive again.

Graph the derivative function so that the points match the relevant points on the original function.

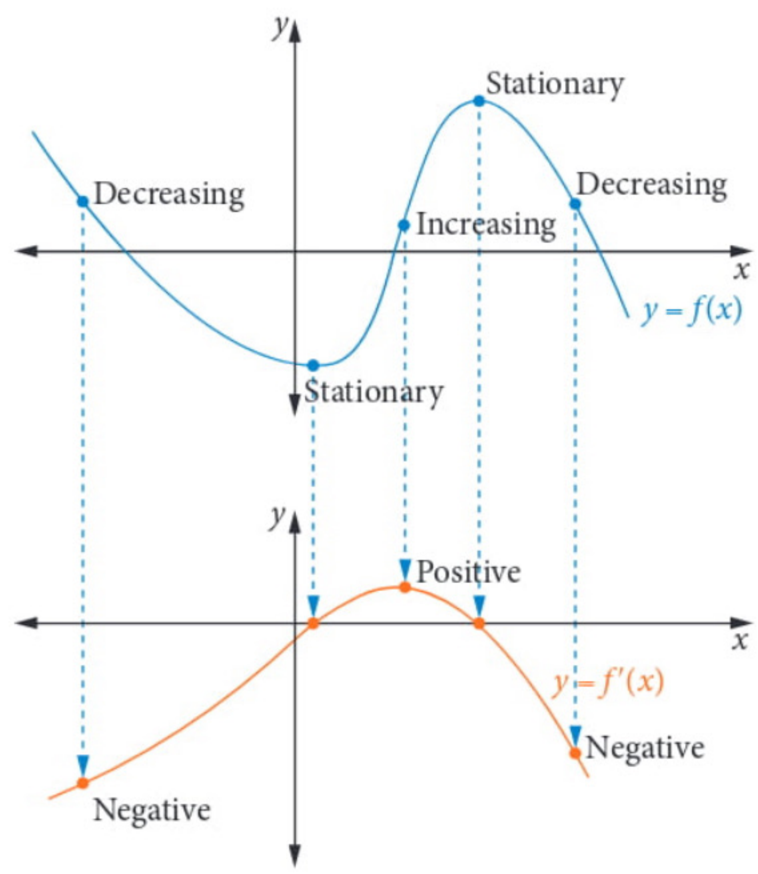
Working



The graphs of $f(x)$ and $f'(x)$

Graph of $f(x)$	increasing	decreasing	stationary	straight line
Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line

The derivative graph may be required when the rule for the original function is not actually given, as shown on the right. In such cases, technology will be less useful and other methods of noticing patterns and relationships must be used.



EXERCISE 2.4 Curve sketching

ANSWERS p. 389

Recap

1 The function $f(x)$ satisfies the following conditions.

- $f'(x) < 0$ where $x < 2$
- $f'(x) = 0$ where $x = 2$
- $f'(x) = 0$ where $x = 4$
- $f'(x) > 0$ where $2 < x < 4$
- $f'(x) > 0$ where $x > 4$

Which one of the following is true?

A The graph of $f(x)$ has a local maximum point at $x = 4$.

B The graph of $f(x)$ has a stationary point of inflection at $x = 4$.

C The graph of $f(x)$ has a local maximum point at $x = 2$.

D The graph of $f(x)$ has a local minimum point at $x = 4$.

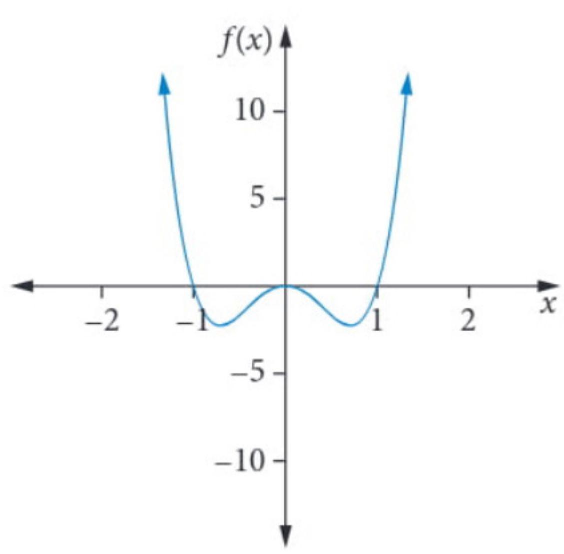
E The graph of $f(x)$ has a stationary point of inflection at $x = 2$.

2 Find any coordinate(s) where the function $f(x) = x^5 - 1$ has a stationary point of inflection.

Mastery

3 WORKED EXAMPLE 6 Sketch the graph of $y = x^3 + 8$, labelling all key features.

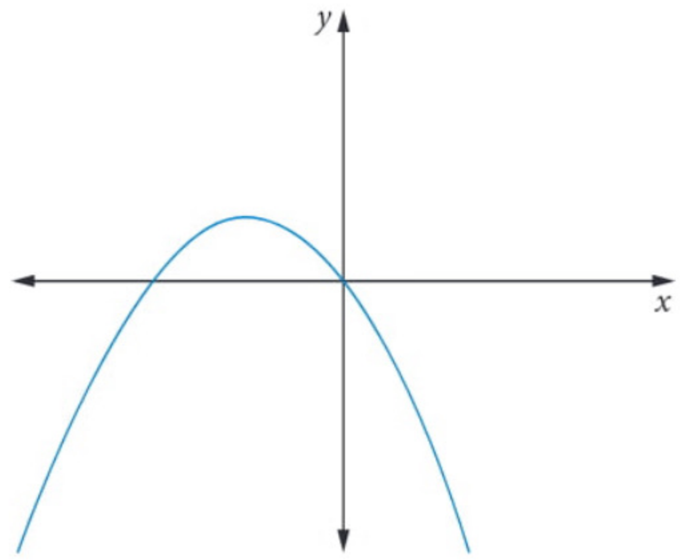
4 WORKED EXAMPLE 7 The graph of $y = f(x)$ is shown below. Sketch the graph of $y = f'(x)$.



5 **a** Sketch the graph of $f(x) = -x(x + 2)(x - 3)$, labelling all key features.

b Sketch the derivative graph of $f(x) = -x(x + 2)(x - 3)$.

6 (2 marks) Given the graph below, sketch the graph of its derivative function.



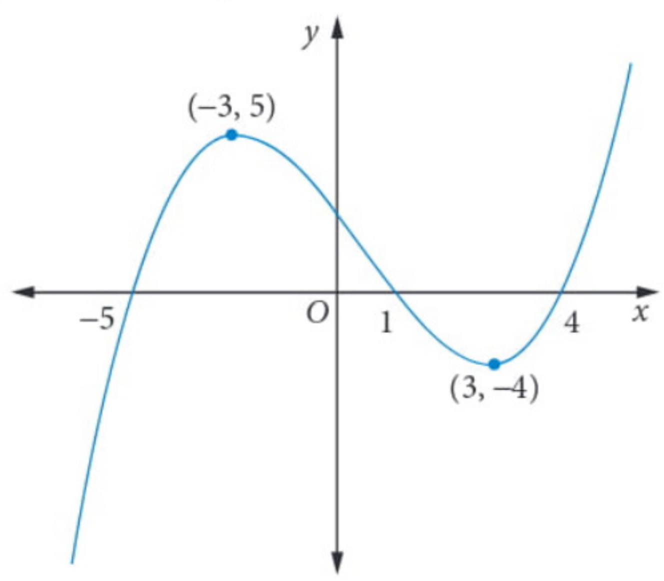
7 (5 marks) A continuous function, f, satisfies the conditions listed below. Using the information, draw a sketch of the function.

- $f(-2) = f'(-2) = f''(-2) = 0$
- $f'\left(\frac{1}{4}\right) = 0, f''\left(\frac{1}{4}\right) = 3$
- Between $-2 < x < \frac{1}{4}$ and $x < -2$, the function has a negative slope.
- $f(x)$ has exactly 2 stationary points.

8 (2 marks) For the graph of $y = f(x)$ shown, state

a the interval(s) when $f'(x)$ is negative (1 mark)

b the coordinate(s) when $f'(x) = 0$ and $f''(x) > 0$. (1 mark)



2.5 Straight line motion

For straight line motion

- **displacement** = $x(t)$ = position of a particle at time t from a chosen origin
- **velocity** = $v(t)$ = velocity at time t, the rate of change of displacement
- **acceleration** = $a(t)$ = acceleration at time t, the rate of change of velocity.

> **Displacement** is a 'signed distance' that can be positive or negative.

> **Velocity** is a 'signed speed' that can be positive or negative.

Hence, velocity is the derivative of displacement, and acceleration is the derivative of velocity (and the second derivative of displacement).

Displacement		Velocity		Acceleration
x	→	$v, \dfrac{dx}{dt}$	→	$a, \dfrac{dv}{dt}, \dfrac{d^2x}{dt^2}$

WORKED EXAMPLE 8 Finding straight line motion

The displacement of a particle travelling in a straight line is given by $x(t) = 2t^3 + t$, with x in metres and t in seconds.

a Find an expression for $v(t)$.

b Find the acceleration of the particle at $t = 3$.

Steps	Working
a Write the function for displacement, then use $\dfrac{dx}{dt}$ to find the velocity.	$x(t) = 2t^3 + t$ $\therefore v(t) = \dfrac{dx}{dt} = 6t^2 + 1$
b **1** Use $\dfrac{dv}{dt}$ to find a.	$v(t) = 6t^2 + 1$ $\therefore a(t) = \dfrac{dv}{dt} = 12t$
2 Find acceleration at $t = 3$.	$a(3) = 36\,\text{m/s}^2$

WORKED EXAMPLE 9 Interpreting straight line motion

A marble dropped into a barrel of water falls such that its height, h metres, after t seconds is given by $h(t) = 1.5 - 0.1t^2$.

a Find the height, velocity and acceleration of the marble at 3 seconds.

b Interpret your answer for the velocity and acceleration in the context of the question.

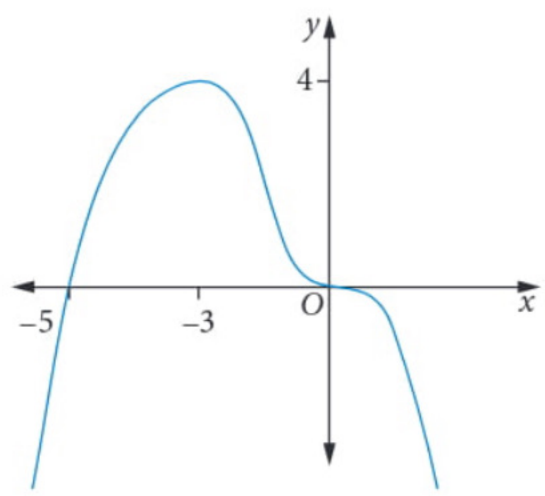
Steps	Working
a **1** Substitute $t = 3$ into $h(t)$ to find the height.	$h(3) = 1.5 - 0.1(3)^2 = 0.6\,\text{m}$
2 Differentiate $h(t)$ and calculate $v(3)$ to find the velocity.	$\dfrac{dh}{dt} = v(t) = -0.2t$ $\therefore v(3) = -0.2 \times 3 = -0.6\,\text{m/s}$
3 Differentiate $v(t)$ and calculate $a(3)$ to find the acceleration.	$\dfrac{d^2h}{dt^2} = \dfrac{dv}{dt} = a(t) = -0.2$ $\therefore a(3) = -0.2\,\text{m/s}^2$
b Comment.	The marble is moving with a speed of 0.6 m/s in a downward direction. The marble is moving with constant acceleration.

Recap

1 Consider $f(x) = x^2 + \dfrac{p}{x}$, $x \neq 0$. There is a stationary point on the graph of f when $x = 2$. The value of p is

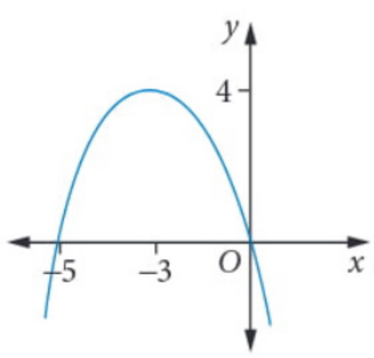
A −16 **B** −8 **C** 2 **D** 8 **E** 16

2 The graph of the function $y = f(x)$ is shown.

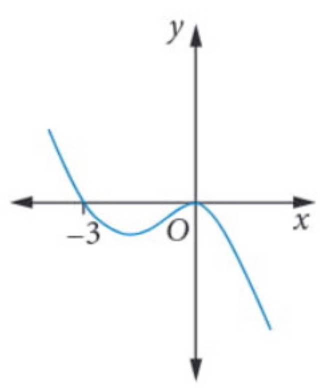


Which of the following could be the graph of the derivative function $y = f'(x)$?

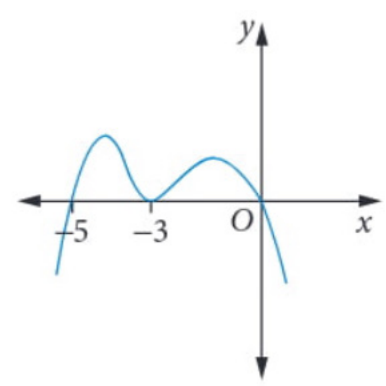
A



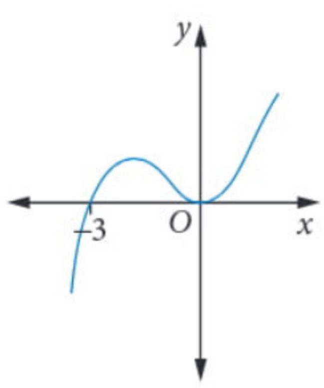
B



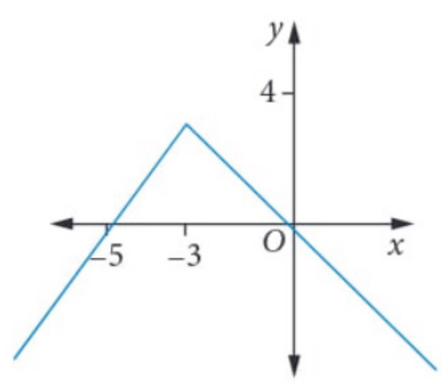
C



D



E



Mastery

3 WORKED EXAMPLE 8 The displacement of a particle travelling in a straight line is given by $x(t) = 2t^2 + 1$, where x is in metres and t is in seconds.

- **a** Find an expression for $v(t)$.
- **b** Find the acceleration of the particle at $t = 2$.

4 WORKED EXAMPLE 9 A ball is thrown into the air such that its height, h metres, after t seconds is given by $h(t) = 1.5t + t^2 - 0.5t^3$ for $0 \leq t \leq 3$.

- **a** Find the height, velocity and acceleration of the ball when $t = 2$ seconds.
- **b** Interpret your answer for the velocity in the context of the question.

5 The displacement of a particle travelling in a straight line is given by $x(t) = 3t^2 + 2t$, where x is in metres and t is in seconds.

- **a** Find an expression for $v(t)$.
- **b** Find the acceleration of the particle at $t = 6$.

Calculator-free

6 (6 marks) An object is travelling along a straight line over time t seconds, with displacement (in metres) according to the equation $x = t^3 + 6t^2 - 2t + 1$.

- **a** Find the equations for its velocity and acceleration. (2 marks)
- **b** What will its displacement be at 5 seconds? (1 mark)
- **c** What will its velocity be at 5 seconds? (1 mark)
- **d** Find the initial acceleration. (1 mark)
- **e** Find its acceleration at 5 seconds. (1 mark)

7 (7 marks) The displacement of an object from a particular point is given by $x(t) = t^3 - 5t^2 + 6t + 10$, where $x(t)$ is in metres and t is in seconds.

- **a** Find the average rate of change in the object's displacement from 2 seconds to 4 seconds. (1 mark)
- **b** Find the velocity at 2 seconds. (1 mark)
- **c** Find the velocity at 4 seconds. (1 mark)
- **d** Find the average of the velocities at 2 seconds and 4 seconds and compare it to the answer for part **a**. (2 marks)
- **e** Find the acceleration at 2 seconds. (1 mark)
- **f** Find the acceleration at 5 seconds. (1 mark)

8 (9 marks) A particle is moving such that its displacement is given by $s = 2t^2 - 8t + 3$, where s is in metres and t is in seconds.

- **a** Find its initial velocity. (1 mark)
- **b** Show that its acceleration is constant and find its value. (2 marks)
- **c** Find the displacement after 5 seconds. (1 mark)
- **d** Determine when the particle will be at rest. (2 marks)
- **e** What will the particle's displacement be at that time? (1 mark)
- **f** Sketch the graph of the displacement against time. (2 marks)

Exam hack

If the particle is at rest, then it is not moving, therefore the velocity is zero.

Calculator-assumed

9 (4 marks) A car is travelling on a day's outing with displacement expressed by $x(t) = 10(t + 1)^2$, where x is the displacement from the driver's home in kilometres and t is the time taken in hours.

- **a** Find its velocity exactly 5 hours from home. (2 marks)
- **b** Find its acceleration exactly 5 hours from home. (2 marks)

10 (2 marks) A toy car is travelling with a velocity expressed by $v(t) = 2t^2 + 1$ m/s. Find its acceleration after travelling for 5 seconds.

11 (2 marks) The velocity of a particle travelling in a straight line is given by $v = (2t - 1)^2$, where v is in m/s and t is in seconds. Determine its acceleration, in m/s^2, at $t = 1$.

12 (4 marks) The position $x(t)$ cm of a particle at time t seconds is given by $x(t) = \frac{t+2}{2t+5}$.

a State the equation for the velocity, $v(t)$, of the particle at time t. (1 mark)

b State the equation for the acceleration, $a(t)$, of the particle at time t. (1 mark)

c Show that the magnitude of the acceleration of the particle is always four times the magnitude of the velocity. (2 marks)

13 (5 marks) A displacement equation is given by $x(t) = pt^2 + qt + r$, where p, q and r are constants and displacement is in metres and time in seconds. Given that $a(3) = -4$, $v(3) = -24$ and $x(3) = -34$, determine the value of p, q and r.

2.6

2.6 Optimisation problems

One common application of differentiation is to solve problems involving maximising or minimising a quantity that can be described by a function. These are sometimes called **optimisation problems**. To solve these problems, it is important to follow a planned process.

Video playlist Optimisation problems

Worksheets Starting maxima and minima problems

Greatest and least values

Applications of optimisation

Optimisation problems

Process for solving optimisation problems

1 Read the question, taking note of words such as maximum/minimum, least/highest, largest/smallest.

2 Set up a function to describe what needs to be maximised or minimised.

3 Make sure the quantity to be maximised or minimised is in terms of one variable only. (If not, rearrange variables in terms of each other.)

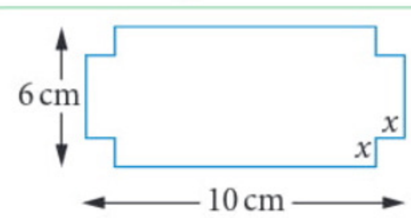
4 Differentiate the function and solve $f'(x) = 0$.

5 You may need to justify whether your answer is the maximum or minimum. This can be done either by using the second derivative or testing the gradient on either side of the stationary point.

6 Re-read the question to make sure you have answered what is asked.

WORKED EXAMPLE 10 Solving optimisation problems

A rectangular piece of cardboard, measuring 10 cm by 6 cm, has squares of length x cm cut out at each corner. The remaining faces are folded up to make a small packing box for jewellery. Find the maximum possible volume, in cm^3, for these jewellery boxes. Give the answer correct to one decimal place.

Steps	Working
1 Sketch a diagram.	
2 Set up an equation to describe the volume of the box.	length = $10 - 2x$, width = $6 - 2x$, height = x $V(x) = x(10 - 2x)(6 - 2x)$ $= 4x^3 - 32x^2 + 60x$
3 Solve $V'(x) = 0$ for local maximum and minimum values.	$V'(x) = 12x^2 - 64x + 60 = 0$ $x = \frac{8 \pm \sqrt{19}}{3}$ ($x = 4.1196$ or $x = 1.2137$) (x cannot be 4.11 cm since the width of the box is only 6 cm.)

4 Use the second derivative to test that the value gives a maximum.

$V''(x) = 24x - 64$

$V''\left(\frac{8-\sqrt{19}}{3}\right) < 0$, maximum

Exam hack

After solving maximum and minimum problems, re-read the question to make sure that you have answered it. One common mistake is to forget to answer the question, even though all the working is correct. Do you need the x or the V value?

5 Substitute this value of x into $V(x)$ to find the maximum volume of the jewellery box.

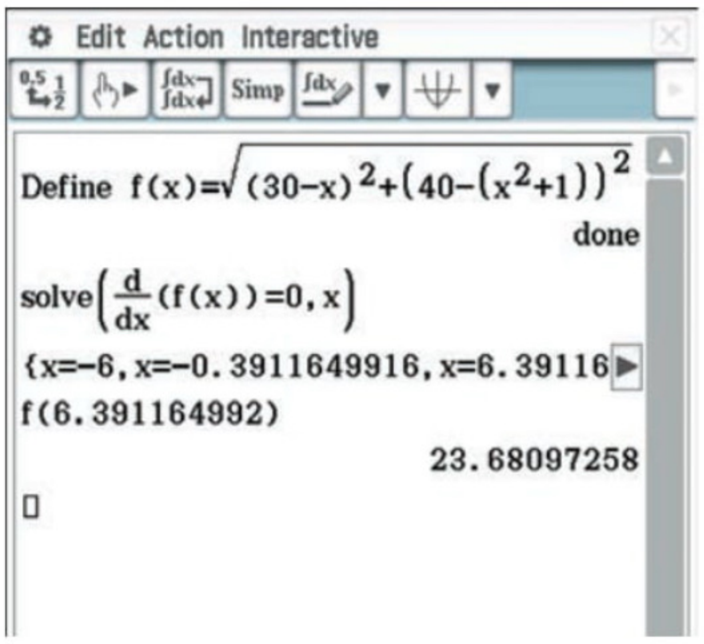
$V(1.2137) = 32.835$

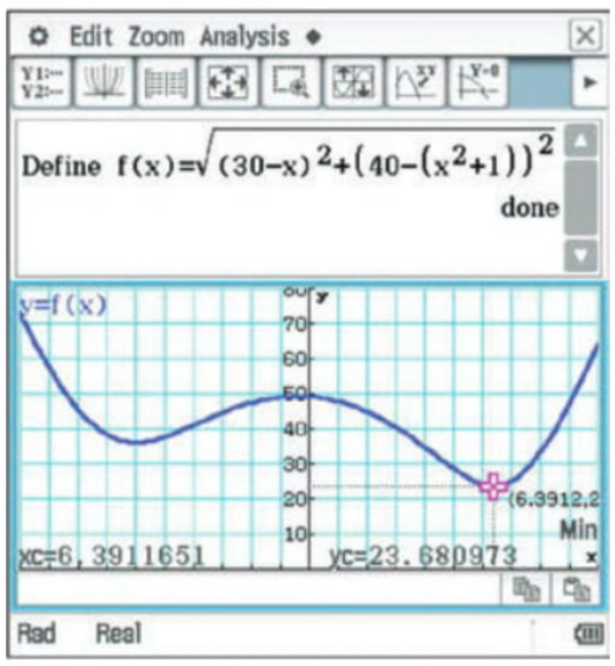
The maximum possible volume of box is 32.8 cm^3.

USING CAS 4 Solving optimisation problems

Find the minimum distance from a point on the graph of $y = x^2 + 1$ to the point (30, 40), correct to two decimal places.

ClassPad

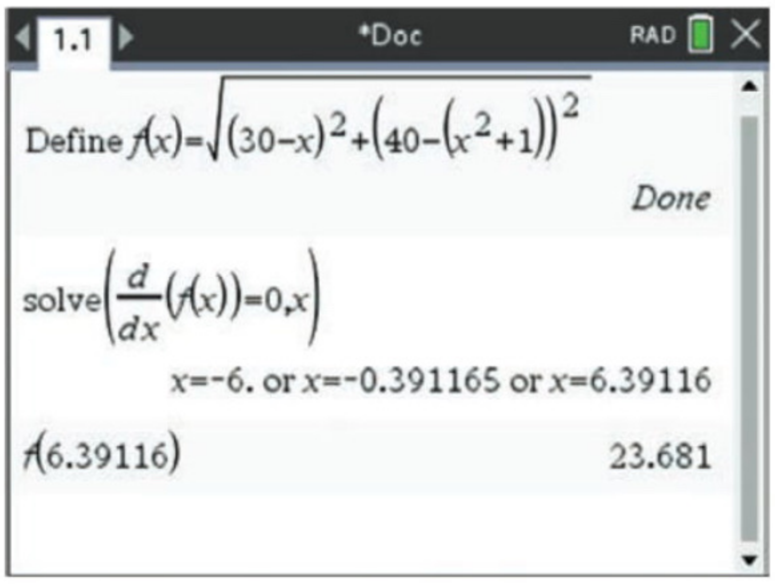


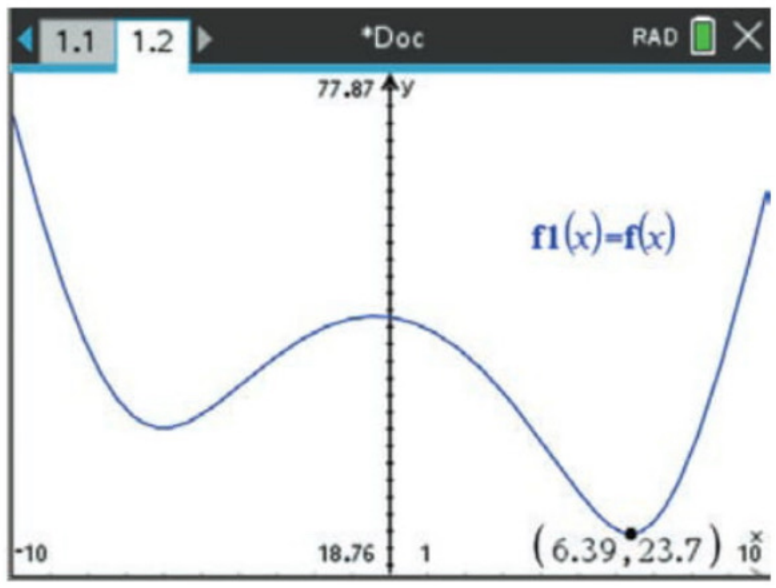


1. Define **f(x)** as shown above.
2. Derive using **Interactive > Calculation > Diff**, then using **Equation/Inequality** solve for **x**.
3. Copy the last solution and calculate **f(6.391164992)**.
4. The minimum distance will be displayed.
5. Graph **f(x)**.
6. Adjust the window settings so that **ymax = 80** to view the local minimums and maximum.
7. Tap **Analysis > G-Solve > Min**.
8. Press the **right arrow** key to move the cursor to the second local minimum.
9. The coordinates of the local minimum will be displayed.

The minimum distance from the function $y = x^2 + 1$ to the point (30, 40) is 23.68.

TI-Nspire





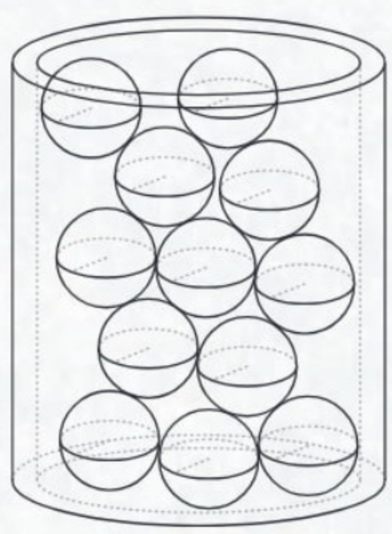
1. Define **f(x)** as shown above.
2. Use the **derivative** template to set the derivative of **f(x) = 0** and solve for **x**.
3. Press **ctrl + enter** for the approximate solutions.
4. Copy the last solution and calculate **f(6.39116)**.
5. The minimum distance will be displayed.
6. Graph **f(x)**.
7. Press **menu > Window / Zoom > Zoom – Fit** to view the local minimums and maximum.
8. Press **menu > Analyze Graph > Minimum**.
9. When prompted for the **lower bound**, click to the left of the second local minimum.
10. When prompted for the **upper bound**, click to the right of the minimum point.
11. The coordinates of the local minimum will be displayed.

The minimum distance from the function $y = x^2 + 1$ to the point $(30, 40)$ is 23.68.

WACE QUESTION ANALYSIS

© SCSA | MM2019 Q16 | Calculator-assumed (10 marks)

A cylindrical glass vase is filled with 20 spherical Christmas decorations as shown below (not all the decorations are visible). All the decorations have a diameter of one-third the internal diameter of the vase and they are completely contained within the vase. For design purposes the sum of the internal diameter of the base of the vase and the vase's internal height is to be 42 cm.



a Show that the volume of unused space in the vase, V, can be expressed as a function of the internal radius of the vase, r, and is given below as (3 marks)

$$V(r) = 2\pi\left(21r^2 - \frac{121}{81}r^3\right).$$

b Use calculus to determine the dimensions of the vase that will maximise the unused space in it. Give your answers rounded to the nearest millimetre. (4 marks)

c Can more than 20 of the spherical decorations fit inside the vase in part **b**? Use calculations to verify your answer. (3 marks)

Video WACE question analysis: Applications of differentiation

2.6

Reading the question

- You should highlight the fact that the question is about cylinders and spheres, and note any relevant formula on the formula sheet.
- Note that the question gives you extra information about the diameter of the balls, and the relationship between the base and height of the vase. Note that the question uses diameter but the formulas are in terms of radius.
- Part **a** asks you to 'show', which means you need enough working to show how you obtained the answer.
- In part **b** the question is asking you to use calculus, which means you must show some differentiation. Part **b** is also asking for the dimensions of the vase, not the volume. Note that the answer is to be rounded to a particular level of accuracy.
- Part **c** is asking for calculations; just answering 'yes' or 'no' will not suffice.

Thinking about the question

- As each part of the question is worth more than 2 marks, you will need to show some working for each part.
- Even when the question says to 'show' or 'use calculus', you can still use CAS to complete steps such as differentiating. Write down each of these steps as part of your working.
- Note that the question is set up so that even if you cannot complete part **a**, you will be able to use that formula for part **b**.
- For part **b**, you should determine the second derivative to check that your answer is a maximum.

Worked solution (✓ = 1 mark)

a $2r + h = 42$

$$h = 42 - 2r$$

$$V(r,h) = \pi r^2 h - 20\left(\frac{4}{3}\pi\left(\frac{r}{3}\right)^3\right)$$

$$V(r) = \pi r^2(42 - 2r) - \frac{80\pi}{81}r^3$$

$$= 2\pi\left(21r^2 - r^3 - \frac{40}{81}r^3\right)$$

$$= 2\pi\left(21r^2 - \frac{121}{81}r^3\right)$$

determines an expression for h in terms of r ✓

states an expression for the volume of unused space in terms of r and h ✓

clearly shows that the expression for h in terms of r can substitute into V and simplifies to determine required result ✓

b $V'(r) = 2\pi\left(42r - \dfrac{363r^2}{81}\right)$

$0 = 42r - \dfrac{363r^2}{81}$

$0 = r\left(42 - \dfrac{363r}{81}\right)$

$r = 0$ or $\dfrac{1134}{121}$ (= 9.372 (3 d.p.))

$V''(r) = 2\pi\left(42 - \dfrac{726r}{81}\right)$

$V''(9.372) = -ve\ (= -84\pi) \Rightarrow \max$

Dimensions are $r = 9.4$ cm and $h = 23.3$ cm.

determines first derivative of $V(r)$ ✓

equates to zero and determines 0 and 9.4 are solutions ✓

clearly shows the use of the second derivative or sign test to show that $r = 9.4$ is a maximum ✓

states the dimensions of the vase that maximise the unused space rounded to the nearest mm ✓

c $V(9.4) = 3863.1\ \text{cm}^3$

$V(\text{decoration}) = 127.7\ \text{cm}^3$

There is likely space for more decorations, but it is not certain as it would depend on the way the balls were packed into the vase.

states the volume of unused space and the volume of one decoration ✓

infers likely to fit more ✓

states the limitation of packing ✓

EXERCISE 2.6 Optimisation problems

ANSWERS p. 390

Recap

1 If the displacement of an object is given by $x(t) = \dfrac{t+1}{1-2t^2}$, determine the acceleration when $t = 2$. Give your answer correct to two decimal places.

2 If the displacement of an object is given by $x(t) = 2(t-3)(t+3)(t+3)$ metres in t seconds, determine when the object is at rest.

Mastery

3 WORKED EXAMPLE 10 A right circular cone is made with height h cm and radius r cm, where the height is 2 cm minus the radius.

Find the maximum volume, in cm^3, of this cone. Give your answer correct to two decimal places.

4 Using CAS 4 Find the minimum distance from the origin to a point on the hyperbola with the rule $y = \dfrac{3}{x-1} + 2$, correct to two decimal places.

Calculator-free

5 (4 marks) A rectangular prism has dimensions 4 cm, x cm and $(4 - x)$ cm. Find the maximum possible volume of this rectangular prism.

6 (5 marks) A wire of length P cm is used to construct a square-based rectangular prism of volume $V\ \text{cm}^3$. Show that the maximum volume occurs when the shape is a cube.

7 (4 marks) A wire frame in the shape of a rectangular box is constructed using P cm of wire. The frame has length x cm, width y cm and height h cm.

a Express the volume, V, of the box in the form $V = axy(P - bx - cy)$, where a, b, c are constants. (2 marks)

b If $y = 2x$, show that the maximum volume is $\frac{P^3}{6 \times 18^2}\text{cm}^3$. (2 marks)

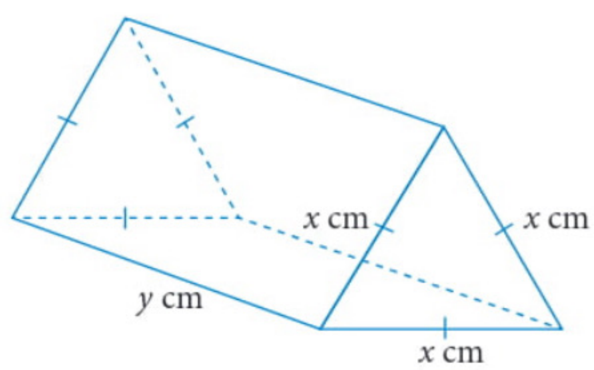
8 (4 marks) Let x and y be two non-negative numbers and let S be the sum of their squares.

a Show that the product of the two numbers can be found using $x\sqrt{S - x^2}$. (2 marks)

b Hence show that the maximum product occurs when $x = \sqrt{\frac{S}{2}}$. (2 marks)

Calculator-assumed

9 (7 marks) A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

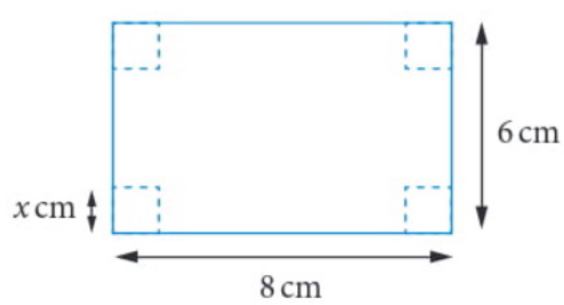
a Find an expression for y in terms of x. (2 marks)

b Show that the total surface area, $A\text{ cm}^2$, of the brick is given by

$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$ (2 marks)

c Find the value of x for which the brick has the minimum total surface area. (You do not have to find this minimum.) (3 marks)

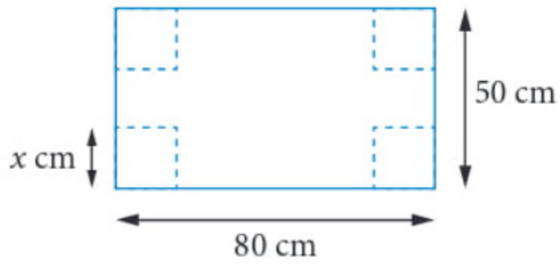
10 (3 marks) Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below.



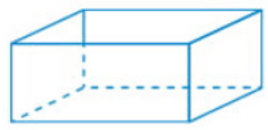
Zoe turns up the sides to form an open box.

Determine the value of x for which the volume of the box is a maximum.

11 (4 marks) A rectangular sheet of cardboard has a length of 80 cm and a width of 50 cm. Squares, of side length x centimetres, are cut from each of the corners, as shown in the diagram.

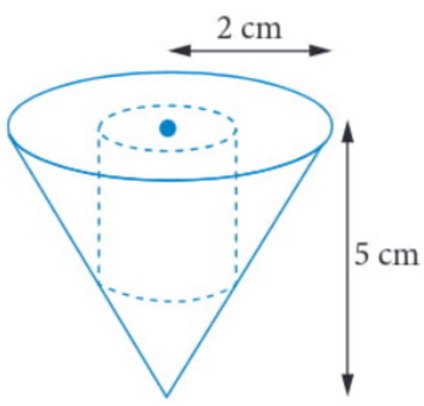


A rectangular box with an open top is then constructed, as shown in the diagram below.



Determine when the volume of the box is a maximum.

12 (5 marks) A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane, as shown in the diagram. The height of the cone is 5 cm and the radius of the cone is 2 cm.

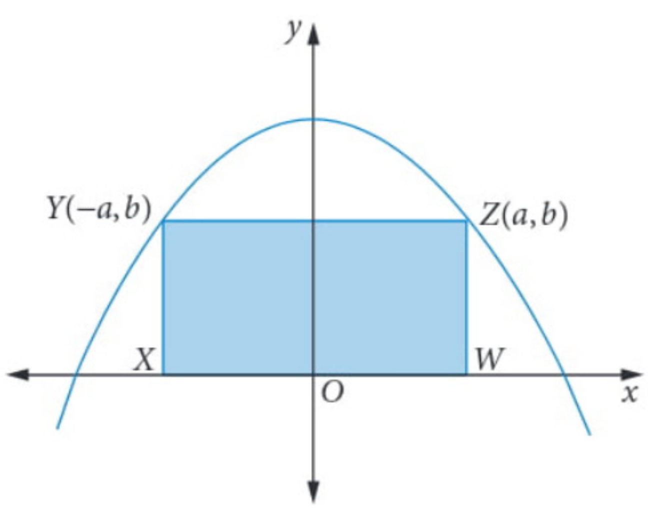


The radius of the cylinder is r cm and the height of the cylinder is h cm, and $h = \dfrac{10 - 5r}{2}$ for the cylinder inscribed in the cone as shown.

a Write a formula for the total surface area (S) of the cylinder in terms of r. (2 marks)

b Find the value of r for which S is a maximum. (3 marks)

13 (4 marks) A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram. The coordinates of Z are (a, b) where a and b are positive real numbers.

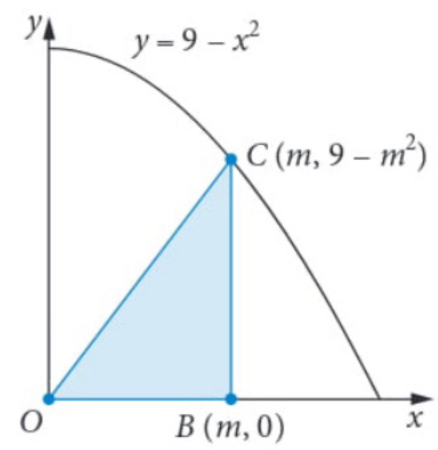


a Find the area, A, of rectangle $XYZW$ in terms of a. (1 mark)

b Find the maximum value of A and the value of a for which this occurs. (3 marks)

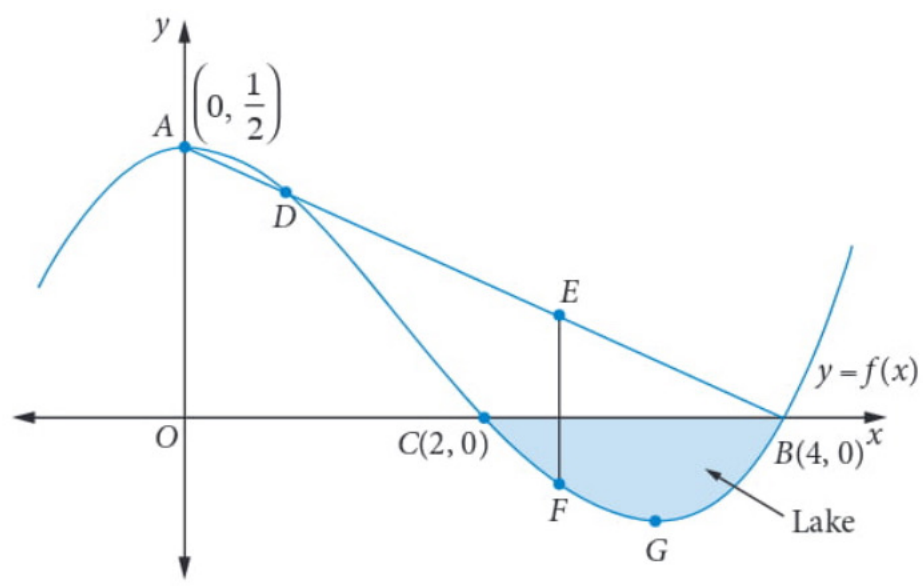
14 (4 marks) P is the point on the line $2x + y - 10 = 0$ such that the length of OP, the line segment from the origin O to P, is a minimum. Find the coordinates of P **and** this minimum length.

15 (3 marks) A right-angled triangle, OBC, is formed using the horizontal axis and the point $C(m, 9 - m^2)$, where $0 < m < 3$, on the parabola $y = 9 - x^2$, as shown below.



Determine the maximum area of the triangle OBC.

16 (13 marks) Tasmania Jones is in Switzerland. He is working as a construction engineer and he is developing a thrilling train ride in the mountains. He chooses a region of a mountain landscape, the cross-section of which is shown in the diagram.



The cross-section of the mountain and the valley shown in the diagram (including a lake bed) is modelled by the function with rule

$$f(x) = \frac{3x^3}{64} - \frac{7x^2}{32} + \frac{1}{2}$$

Tasmania knows that $A\left(0, \frac{1}{2}\right)$ is the highest point on the mountain and that $C(2, 0)$ and $B(4, 0)$ are the points at the edge of the lake, situated in the valley. All distances are measured in kilometres.

a Show use of calculus to find the exact coordinates of G, the deepest point in the lake. (3 marks)

Tasmania's train ride is made by constructing a straight railway line AB from the top of the mountain, A, to the edge of the lake, B. The section of the railway line from A to D passes through a tunnel in the mountain.

b Write down the equation of the line that passes through A and B. (2 marks)

In order to ensure that the section of the railway line from D to B remains stable, Tasmania constructs vertical columns from the lake bed to the railway line. The column EF is the longest of all possible columns.

c Show use of calculus to find the x-coordinate of E. (3 marks)

Tasmania's train travels down the railway line from A to B. The speed, in km/h, of the train as it moves down the railway line is described by the function

$$v(x) = k\sqrt{x} - mx^2 \text{ for } 0 \le x \le 4$$

where x is the x-coordinate of a point on the front of the train as it moves down the railway line, and k and m are positive real constants.

The train begins its journey at $A\left(0, \frac{1}{2}\right)$. It increases its speed as it travels down the railway line. The train then slows to a stop at $B(4, 0)$, that is, $v(4) = 0$.

d Find k in terms of m. (2 marks)

e Show use of calculus to find the value of x for which the speed, v, is a maximum. (3 marks)

2 Chapter summary

The increments formula

If δx is small, we can say $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$. Rearranging gives $\delta y \approx \frac{dy}{dx} \times \delta x$, which is called the **increments formula**.

Second derivative

The **second derivative** is the rate of change of the first derivative function.

Points of inflection

A non-stationary **point of inflection** is where a function changes **concavity**.

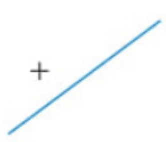
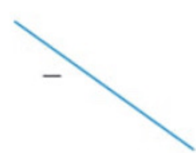
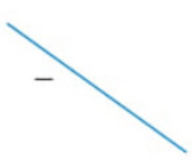
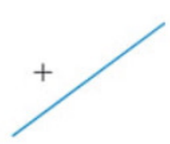
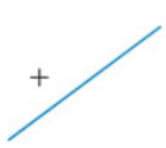
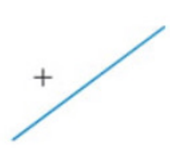
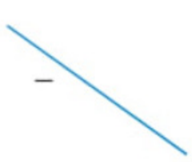
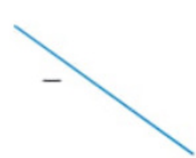
Increasing and decreasing functions

A function is **increasing** when it has a positive gradient $\left(\frac{dy}{dx} > 0\right)$ and its graph is pointing up.

A function is **decreasing** when it has a negative gradient $\left(\frac{dy}{dx} < 0\right)$ and its graph is pointing down.

Stationary points

- A **turning point** (local maximum or minimum point) is found where $f'(x) = 0$ and also where the sign of the gradient changes on either side of the stationary point.
- A **stationary point of inflection** is found where $f'(x) = 0$ and the concavity changes. The sign of the gradient stays *the same* on both sides of the stationary point.
- If $f''(x) > 0$, the turning point is a minimum.
- If $f''(x) < 0$, the turning point is a maximum.
- If $f''(x) = 0$, further investigation into the nature of the stationary point is required.

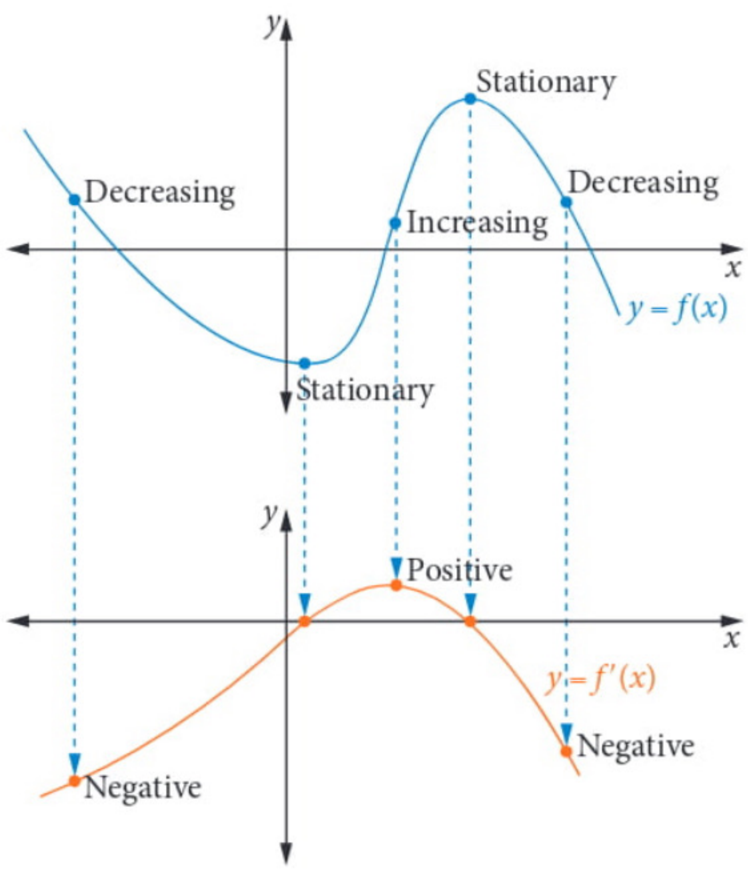
$f'(x)$ before the stationary point	$f'(x)$ at the stationary point	$f'(x)$ after the stationary point	$f''(x)$ at the stationary point	Type of stationary point
	0		Less than zero	Local maximum point 
	0		Greater than zero	Local minimum point 
	0		0	Stationary point of inflection 
	0		0	Stationary point of inflection 

Curve sketching

- Identify **key features** of a graph when sketching:
  - general shape
  - y-intercept
  - x-intercept(s)
  - stationary points, including their nature.

The graph of the derivative function

Graph of $f(x)$	increasing	decreasing	stationary	straight line	discontinuous point, vertical asymptote
Graph of $f'(x)$	positive, above x-axis	negative, below x-axis	0 (x-intercept)	horizontal (flat) line	discontinuous point, asymptote



Straight line motion

- **displacement** = $x(t)$ = position of a particle at time t from a chosen origin
- **velocity** = $v(t)$ = velocity at time t, the rate of change of displacement
- **acceleration** = $a(t)$ = acceleration at time t, the rate of change of velocity

Hence, velocity is the derivative of displacement, and acceleration is the derivative of velocity (and the second derivative of displacement).

Optimisation problems

- Techniques for finding maximum and minimum values to help solve problems
  1. Read the question, taking note of words such as maximum/minimum, least/highest, largest/smallest.
  2. Set up a function to describe what needs to be maximised or minimised.
  3. Make sure the quantity to be maximised or minimised is in terms of one variable only. (If not, rearrange variables in terms of each other.)
  4. Differentiate the function and solve $f'(x) = 0$.
  5. You may need to justify whether your answer is the maximum or minimum. This can be done either by using the second derivative or testing the gradient on either side of the stationary point.
  6. Re-read the question to make sure you have answered what is asked.

Cumulative examination: Calculator-free

Total number of marks: 21 Reading time: 2 minutes Working time: 20 minutes

1 (3 marks) Find the equation of the tangent to $y = x^2(x + 1)$ at the point $(-1, 0)$.

2 (6 marks) For the function $f(x) = x^2(2x - 5)$

a determine $f'\left(\frac{5}{3}\right)$ and $f''\left(\frac{5}{3}\right)$ (4 marks)

b in relation to the graph of $f(x)$, explain the meaning of your answers in part **a**. (2 marks)

3 (4 marks) Use the quotient rule to show that the derivative of $\frac{2x^2 + 1}{\sqrt{x}}$ is equal to $\frac{6x^2 - 1}{2x^{\frac{3}{2}}}$.

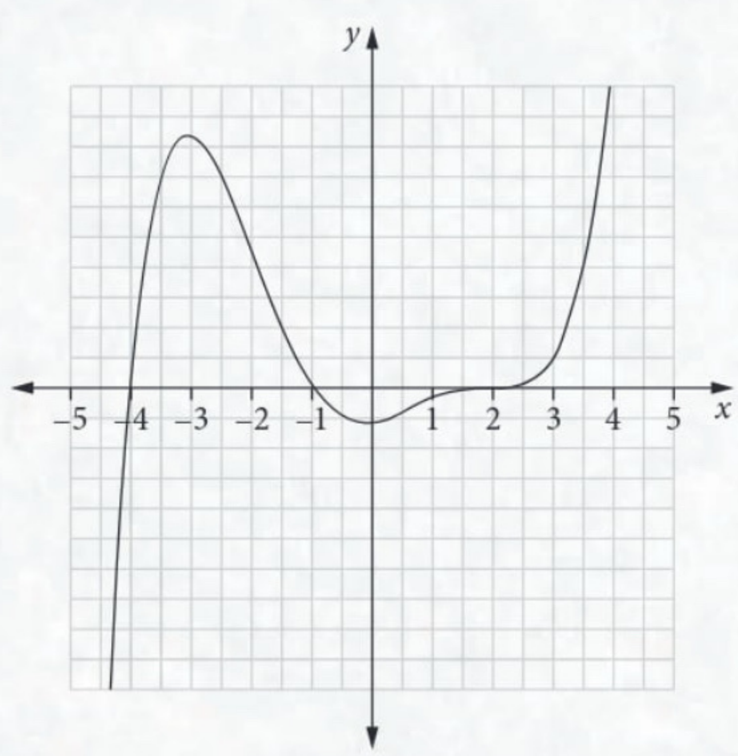
4 (5 marks) A continuous function, f, satisfies the conditions

- $f(-2) = 0$
- for $x < -\frac{5}{4}, f(x) > 0$
- for $-\frac{5}{4} < x < 1, f(x) < 0$
- f has exactly 2 stationary points
- $f(1) = f'(1) = f''(1) = 0$.

Sketch the function.

5 (3 marks) The graph of $y = f(x)$ is drawn below.

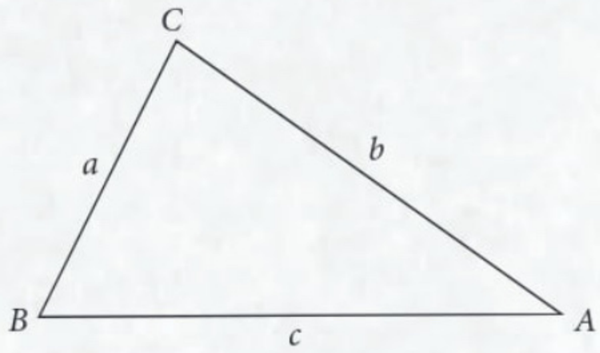
Sketch the graph of $y = f'(x)$.



Cumulative examination: Calculator-assumed

Total number of marks: 30 Reading time: 3 minutes Working time: 30 minutes

1 © SCSA MM2016 Q11 (3 marks) The area of a triangle can be found by the formula: Area $= \dfrac{ab\sin C}{2}$.



Using the increments formula, determine the approximate change in area of an equilateral triangle, with each side of 10 cm, when each side increases by 0.1 cm.

2 © SCSA MM2017 Q17 (6 marks) A beverage company has decided to release a new product. 'Joosilicious' is to be sold in 375 mL cans that are perfectly cylindrical. {Hint: 1 mL = 1 cm^3}

a If the cans have a base radius of x cm, show that the surface area of the can, S, is given by: $S = 2\pi x^2 + \dfrac{750}{x}$. (2 marks)

b Using calculus methods, and showing full reasoning and justification, find the dimensions of the can that will minimise its surface area. (4 marks)

3 © SCSA MM2018 Q15 (5 marks) The population of mosquitos, P (in thousands), in an artificial lake in a housing estate is measured at the beginning of the year. The population after t months is given by the function, $P(t) = t^3 + at^2 + bt + 2$, $0 \le t \le 12$.

The rate of growth of the population is initially increasing. It then slows to be momentarily stationary in mid-winter (at $t = 6$), then continues to increase again in the last half of the year.

Determine the values of a and b.

4 (10 marks) The displacement of a particle s moving along a straight line at time t seconds is given by $s = t^3 - 4t^2 + 4t - 10$ metres.

a Determine the change in displacement in the first 2 seconds. (2 marks)

b Determine the velocity of the particle when $t = 5$ seconds. (2 marks)

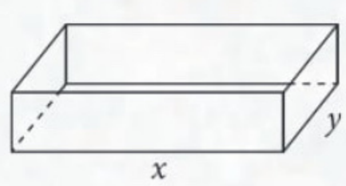
c Determine when the particle is instantaneously at rest. (2 marks)

d Determine the initial acceleration of the particle. (2 marks)

e Determine the distance travelled in the first 2 seconds. (2 marks)

5 (6 marks) An **open** (no lid) rectangular tank of volume 8 cm^3 has length x cm and width y cm.

a Show that the surface area of the tank can be written in the form: $A = xy + \dfrac{16}{x} + \dfrac{16}{y}$. (2 marks)



b Due to storage requirements, the length, x, is a fixed constant, $x = k$. The minimum value of A can be expressed in the form: $A = a\sqrt{k} + \dfrac{b}{k}$. Determine the values of the constants a and b. (4 marks)

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